Free Vibration of Neutrally Buoyant Inflatable Cantilevers in the Ocean Environment

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Free vibration analysis of the neutrally buoyant inflated cantilevers, made of plastic sandwiched films, is presented, accounting for the added inertia and nonlinear hydrodynamic drag. The significant feature of the analysis is the reduction of the shell equations (the membrane, Flügge's, and Herrmann-Armenákas') into a single equation which is similar in form to that for a vibrating beam with rotary inertia effects. The natural frequencies obtained are compared with the experimental results and those predicted by the Rayleigh-Ritz method in conjunction with the Washizu and membrane shell theories. The analyses show, and the experimental program confirms, that Flügge's shell equation in reduced form is capable of predicting free vibration behavior quite accurately. However, the reduction technique should be applied with care, since in several cases it leads to misleading results, e.g., in the case of the Herrmann-Armenákas theory, generally accepted to be one of the most elaborate.

Nomenclature

A_2	= coefficient, $(1-\nu^2)^3P^3/4+(13+7\nu)(1-\nu^2)^2P^2/8+(14-2\nu-12\nu^2)(1-\nu)P/8+(1-\nu)(1-\nu^2)/2$
A_4	= coefficient, $G[2(1-\nu^2)^2P^2+(3-\nu)(1-\nu^2)P+(1-\nu)]$
A_5	= coefficient, $G[2(1-\nu^2)^2P^2-(11-2\nu)(1-\nu^2)P/2-5/2+3\nu/2+\nu^2]$
B_1	$=(L/a)^4B_3A_4/A_2$
B_2	$=(L/a)^2A_5B_3/A_2$
B_3	$=E(a/L)^4/\rho a^2[2+(1+C_m)(\rho_w/\rho)(a/h)]$
C_i	= coefficients in solution for zero drag, Eqs. (4) and (6)
C_d, C_m	= drag and added inertia coefficients, respectively
E	= Young's modulus
G	= parameter, $pa^2(1-\nu^2)/E$
K_r	= normalizing multiplier, Eq. (7)
L	= length of cantilever beam
P	= dimensionless pressure, pa/Eh
T	= kinetic energy of shell
U	= potential energy of shell
a	= radius of beam
d	= diameter of beam
e_x, e_θ	= axial and circumferential strains of a shell section, respectively, Eqs. (19)
h	= thickness of beam
p	= internal pressure
t	= time
$\bar{t}$	= time variable, Eq. (11)
v	= transverse velocity of a beam
w	= flexural displacement of beam
x, y, z	= orthogonal displacement components in axial, circumferential, and radial directions, respectively, Fig. 2
$\bar{x}$	= axial coordinate for cylindrical shell, Fig. 2
$\bar{z}$	= distance from the middle surface of a shell section
$\Phi_i(\xi)$	= eigenfunctions of a cantilever
$\Psi_i(\xi)$	= eigenfunctions of a clamped-pinned beam
Ω	= dimensionless frequency, Eq. (25b)
α	= damping parameter of beam
β	= mode shape of zero drag solution, Eq. (4b)
β_{irs}	= constant, $\int_0^I \Phi_i(\xi)\Phi_r(\xi)\Phi_s(\xi)\,d\xi$
$\epsilon_x, \epsilon_\theta$	= axial and circumferential middle surface strains, respectively
η	= dimensionless transverse displacement, w/d
θ	= circumferential coordinate, Fig. 2
κ_x, κ_θ	= axial and circumferential curvatures, respectively, Eqs. (21)
$\lambda, \lambda', \lambda''$	= solutions to frequency Eqs. (5)
ν	= Poisson's ratio
ξ	= dimensionless distance from root (clamped end) of a cantilever
ρ, ρ_w	= mass densities of tube wall and water, respectively
$\sigma_x, \sigma_\theta, \sigma_{x\theta}$	= vibratory stresses, Eqs. (16)
$\sigma_x^i, \sigma_\theta^i, \sigma_{x\theta}^i$	= initial stresses due to internal pressure, Eq. (18)
τ	= dimensionless time, $\sqrt{B_3}t$

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I. Introduction

A. Preliminary Remarks

CONSIDERABLE attention has been given lately to the behavior of thin inflatable shells. Besides their abilities to resist loads efficiently by normal tensile stresses, inflatable shells have the advantages of being lightweight, compact, and collapsible, implying ease in transportation and erection for services. The state-of-the-art in inflatable shell research is summarized by Leonard.[1] Inflatable shells have already exhibited their potential in the design of structural components for aerospace and oceanographic systems. Brauer[2] has discussed in considerable detail, which includes design and performance data, a variety of inflatable structures having aerospace application. On the other hand, neutrally buoyant inflated structures have been proposed for underwater applications like submarine detection, oceanographic survey, lifting surfaces of hydrofoils, etc.

Of particular interest is the use of sonobuoys in submarine detection systems. Sonobuoys are passive listening devices conventionally housed in a cylinder about 0.9-m (3-ft) long and 12.7 to 15.3 cm (5 to 6 in.) in diameter. The containers are deposited from an aircraft in the area of interest. Upon hitting the water, a hydrophone attached by a cable to the floating container is released. All acoustic signals received at the hydrophone are relayed back to the aircraft via a transmitter. Since the target emits signals on an unknown time base, at least three or four hydrophones are needed to locate it in two or three dimensions, respectively.

The sonobuoy has a preset lifetime after which it turns itself off and sinks. Extensive research has established that the efficiency of this operation can be improved by using an array of inflatable tubes, each carrying a hydrophone at one end and joined to a pump-equipped central head at the other (Fig. 1). The target can then be located through processing of signals received by the array, provided that the position and orientation of the array are known.

The optimal design of such a submarine detection system requires a knowledge of its response to in-service loads such as ocean currents, waves, and other local disturbances. The general motion of the system is quite complex because of the large number of degrees of freedom involved.[3] Hence, to achieve better appreciation of the intricate dynamical interactions, it seems appropriate to study constituent subassemblies separately. The array arm being the most significant component from dynamical considerations, the attention is focussed on it in this investigation.

B. Literature Review

A knowledge of the hydrodynamic forces acting on a vibrating cylinder is essential to the study of its underwater dynamics. A number of model and prototype investigations devoted to force data is summarized by Wiegel.[4] The conventional Morison's-type equation derived independently by Morison et al.[5] and Keulegan and Carpenter,[6] assumes that the total hydrodynamic force can be obtained by linearly adding the drag and added inertia components, and is virtually universally used in this class of investigations. Keulegan and Carpenter investigated the drag and inertia coefficients of cylinders in simple sinusoidal currents and observed that they show opposite behaviors over the range of the period parameter considered. However, the sum of the two forces deviates relatively little from the average value. Stelson and Mavis[7] studied the virtual mass of long circular cylinders oscillating in water and found that for cylinders with large length to diameter ratios the added mass approaches unity, as predicted by potential flow analysis. Jen[8] observed, experimentally, that the forces exerted by uniform periodic waves in relatively deep water give an average added mass coefficient of 1.04. Laird et al.[9] and Toebes et al.[10] assumed a constant mass coefficient and included all its deviations from unity in the variation of the drag coefficient. Although the drag coefficient was found to change, the variations were not substantial, provided that the vibrational frequency was not close to the Strouhal frequency. Using a similar concept, Protos et al.[11] also considered a fixed apparent mass and studied the variation of the remaining force with the frequency ratio (ratio of the natural frequency of the cylinder to the Strouhal frequency). In another study, Laird et al.[12] demonstrated that wave forces on a circular cylinder could be influenced significantly by eddy-shedding from neighboring cylinders.

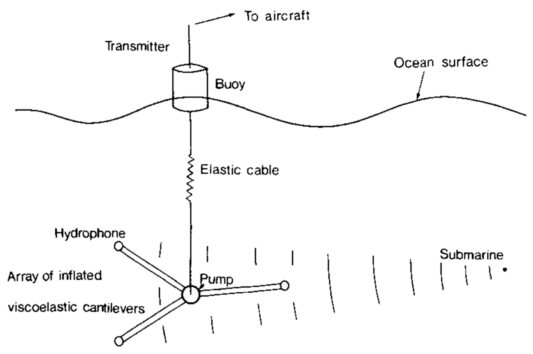

Fig. 1 Schematic diagram of a submarine detection system using an array of neutrally buoyant inflatable cantilevers.

In contrast to the extensive literature on apparent mass effects for a rigid cylinder in a fluid, the corresponding studies for a flexible cylinder are relatively scarce. Landweber[13,14] and Warnock[15] investigated dynamics of an elastic cylinder in an incompressible, inviscid fluid to determine the apparent mass effects. However, the potential flow assumed discounted any hydrodynamic damping forces. The flexural vibration of an inflated cylindrical cantilever in air has been studied by Douglas[16] and Corneliussen and Sheild.[17] Misra and Modi[18] on the other hand, investigated an inflated viscoelastic cantilever vibrating in water. In the study, a detailed analysis of the cylindrical cantilever in the presence of hydrodynamic drag and a tensile follower force was given. However, the elementary beam theory employed does not account for the circumferential stress induced by the internal pressure.

To account fully for the stresses arising due to internal pressure, one has to resort to thin shell theory. However, only a small portion of the vast amount of available literature on shell vibration is concerned with the beam-bending mode of interest here. The reason may be that, as pointed out by Forsberg,[19] for relatively long shells without internal pressure, the beam-bending mode analysis can be considerably simplified without much loss of accuracy by considering the shell as a thin-walled beam and applying the beam theory. Kornecki[20] has shown that, for the beam-bending mode of shells without internal pressure, the Goldenveizer shell equations reduce to an equation very similar to the one for the transverse vibrations of a beam with rotary inertia included.

Incorporation of initial stress effects due to internal pressure requires a generalization of the equations of motion for thin shells. Because of the relative mathematical simplicity, the vast majority of studies made to date have dealt with shells having their boundaries supported by "Shear Diaphragms" (SD). Straightforward methods for handling other edge conditions, including an exact procedure, are available but have been sparingly applied because of the great deal of effort required. For other boundary conditions, the problem is considerably more complicated and relatively few results are available. Modi and Poon,[21] using the procedure outlined by Kornecki,[20] showed that, for the beam-bending mode of inflated long shells, the Flügge's shell equations may also be reduced to a single equation similar in form to that for the transverse vibration of a beam with rotary inertia included. The Rayleigh-Ritz method or its equivalent has been used by several investigators to study the motion of shells with various boundary conditions. Sewall and Naumann[22] used the Rayleigh-Ritz technique with beam functions and the Goldenveizer-Novozhilov shell theory to obtain frequencies for clamped-free shells and compared them with experimental results. They employed seven terms in the assumed mode shapes to obtain convergence of the Ritz procedure. Results were also obtained by Resnick and Dugundji[23] using an energy method equivalent to Rayleigh-Ritz, beam functions, and the Sanders shell theory. Good agreement between theory and experiment was found only for modes with more than five circumferential waves. Extensive numerical results for clamped-free shells were obtained by Sharma and Johns[24-26] using the Ritz method and the Flügge shell equations. Displacement functions were assumed to be a combination of the clamped-free and clamped-pinned beam functions. It may be pointed out that the foregoing approximate investigations using the Rayleigh-Ritz procedure were confined to the case of zero initial stress.

C. Scope of the Investigation

In the dynamical study of any system, a knowledge of the response characteristics of the constituent members necessarily forms a prerequisite. With this in mind, and because of the complex nature of the problem, a small subsystem is selected. The objective here is to study dynamical behavior of the neutrally buoyant, inflated, cylindrical beams, constructed from sandwiched plastic films, forming the hydrophone array. Shell equations are used to incorporate the initial stresses due to internal pressure. The governing reduced equations for three shell theories are studied, namely, the membrane, Flügge's, and Hermann and Armenákas'. The natural frequencies obtained are compared with the experimental results and those predicted by the Rayleigh-Ritz method in conjunction with the Washizu and membrane shell theories.

The presence of hydrodynamic forces introduces nonlinearities into the governing equation of motion. Effect of this nonlinear drag force on the free response is studied using the perturbation technique on the reduced Flügge equation.

II. Free Vibration Analysis

A. Reduced Shell Equation Approach

Formulation

The reduced shell equation for the beam-bending mode (Figs. 2 and 3) obtained from Flügge's theory has been shown previously[21] to be

$$\frac{\partial^4\eta}{\partial\xi^4}+B_2\frac{\partial^4\eta}{\partial\xi^2\partial\tau^2}+B_1\frac{\partial^2\eta}{\partial\tau^2}+\alpha\frac{\partial\eta}{\partial\tau}\left|\frac{\partial\eta}{\partial\tau}\right|=0 \tag{1a}$$

where

$$\eta=w/d,\ \tau=\sqrt{B_3}t$$

and

$$\alpha\simeq\frac{2C_d}{\pi(1+C_m)} \tag{1b}$$

The boundary conditions are given by

$$\eta(0,\tau)=\frac{\partial\eta(0,\tau)}{\partial\xi}=\frac{\partial^2\eta(1,\tau)}{\partial\xi^2}=\frac{\partial^3\eta(1,\tau)}{\partial\xi^3}=0 \tag{1c}$$

This nonlinear partial differential equation does not seem to have any known closed-form solution. Hence, one is forced to approach the problem through numerical analysis or approximate procedures.

The same procedure has also been applied by Poon[27] to reduce the membrane[28,29] and the Hermann-Armenákas'[28,30] theories, respectively, to one single governing equation similar in form to that of Eq. (1a).

Solution for Zero Drag

For determination of the natural frequency, it can be assumed, at least up to the first-order approximation, that the period of oscillation is essentially unaffected by the presence of hydrodynamic drag.[3] Dropping the drag term, Eq. (1a) becomes

$$\frac{\partial^4\eta}{\partial\xi^4}+B_2\frac{\partial^4\eta}{\partial\xi^2\partial\tau^2}+B_1\frac{\partial^2\eta}{\partial\tau^2}=0 \tag{2}$$

Assuming solution of the form

$$\eta(\xi,\tau)=\beta(\xi)f(\tau) \tag{3}$$

and substituting into Eq. (2) gives

$$f=f_c\cos(\lambda^2\tau)+f_s\sin(\lambda^2\tau) \tag{4a}$$

$$\beta=C_1\cosh(\lambda'\xi)+C_2\sinh(\lambda'\xi)+C_3\cos(\lambda''\xi)+C_4\sin(\lambda''\xi) \tag{4b}$$

Applying the boundary conditions, Eq. (1c) gives the following equations relating λ, λ' and λ'':

$$B_2^2\lambda^4+2\sqrt{B_1}(1+\cosh\lambda'\cos\lambda'')-\sqrt{B_1}B_2\lambda^2\sinh\lambda'\sin\lambda''=0 \tag{5a}$$

$$\lambda'^2-\lambda''^2=B_2\lambda^4 \tag{5b}$$

$$\lambda'\lambda''=\sqrt{B_1}\lambda^2 \tag{5c}$$

By solving Eqs. (5), the natural frequencies and the associated mode shapes can be found. The general solution is then given by

$$\eta(\xi,\tau)=\sum_{r=1}^{\infty}\beta_r(\xi)f_r(\tau) \tag{6}$$

where $\beta_r(\xi)$ is the normalized shape given by

$$\beta_r(\xi)=K_r[\cosh\lambda_r'\xi-C_5\sinh\lambda_r'\xi-\cos\lambda_r''\xi+C_6\sin\lambda_r''\xi]$$

with

$$C_5=\frac{\lambda_r'^2\cosh\lambda_r'+\lambda_r''\cos\lambda_r''}{\lambda_r'^2\sinh\lambda_r'+\lambda_r'\lambda_r''\sin\lambda_r''}$$

$$C_6=\frac{\lambda_r'^2\cosh\lambda_r'+\lambda_r''^2\cos\lambda_r''}{\lambda_r'\lambda_r''\sinh\lambda_r'+\lambda_r''^2\sin\lambda_r''}$$

Here, K_r is chosen to normalize $\beta_r(\xi)$, i.e.,

$$\int_0^1\beta_r^2(\xi)\,d\xi=1$$

Analysis showed K_r to be

$$K_r=\Big\{1+\frac{(C_6^2-C_5^2)}{2}+\frac{\sinh\lambda_r'}{2\lambda_r'}[(1+C_5^2)\cosh\lambda_r'$$

$$-2C_5\sinh\lambda_r']+\frac{\sin\lambda_r''}{2\lambda_r''}[(1-C_6^2)\cos\lambda_r''-2C_6\sin\lambda_r'']$$

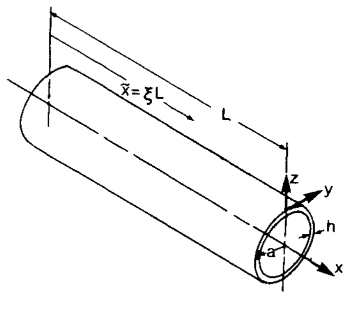

Fig. 2 Geometry and coordinate system for a circular cylindrical shell.

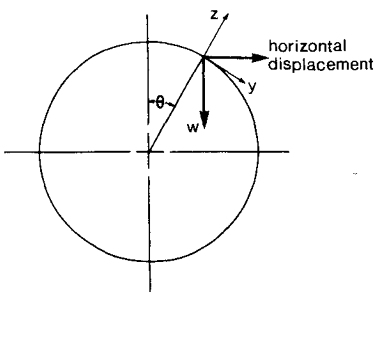

Fig. 3 Vertical and horizontal displacement components.

$$+\frac{2}{\lambda_r'^2+\lambda_r''^2}[(C_5\lambda_r'-C_6\lambda_r'')\cosh\lambda_r'\cos\lambda_r''$$

$$-(\lambda_r''+C_5C_6\lambda_r')\cosh\lambda_r'\sin\lambda_r''+(C_5C_6\lambda_r''-\lambda_r')$$

$$\sinh\lambda_r'\cos\lambda_r''+(C_5\lambda_r''+C_6\lambda_r')\sinh\lambda_r'\sin\lambda_r''$$

$$+(C_5\lambda_r'-C_6\lambda_r'')\Big\}^{-1/2} \tag{7}$$

The natural frequencies are thus given by

$$\omega_r^2=\sqrt{B_3}\,\lambda_r^2 \tag{8}$$

where λ_r is the solution of Eq. (5).

The orthogonality condition for the normalized mode shape $\beta(\xi)$ is found to be[27]

$$\int_0^1 \beta_m\beta_n\,d\xi=\frac{B_2}{B_1}\int_0^1\beta_m'\beta'_n\,d\xi-\frac{B_2}{B_1(\lambda_m^4-\lambda_n^4)}$$

$$\times[\lambda_m^4\beta_m'(1)\beta_n(1)-\lambda_n^4\beta_m(1)\beta_n'(1)]\text{ for } m\neq n \tag{9}$$

Perturbation Solution

To investigate the effect of the hydrodynamic drag on the free response, Eq. (1a) must be solved. The nonlinear nature of the equation does not permit an exact solution and one is forced to resort to approximate analysis. In this section, the perturbation technique is used to study the response of the inflated cantilever subjected to hydrodynamic drag:

$$\frac{\partial^4\eta}{\partial\xi^4}+B_2\frac{\partial^4\eta}{\partial\xi^2\partial\tau^2}+B_1\frac{\partial^2\eta}{\partial\tau^2}\pm\alpha\left(\frac{\partial\eta}{\partial\tau}\right)^2=0 \tag{1a$'$}$$

where the appropriate sign for the drag term is chosen so as to oppose the motion. It is sufficient to solve the preceding equation either for positive or negative sign over half a cycle; solution for the other half is obtained simply by reversing the sign of α with new initial conditions.

The solution for the negative sign is sought in the form

$$\eta(\xi,\tau)=\eta_0(\xi,\tau)+\alpha\eta_1(\xi,\tau)+\alpha^2\eta_2(\xi,\tau)+\ldots \tag{10}$$

A new time variable $\tilde{t}$ defined by

$$\tilde{t}=\tau[1+\alpha b_1+\alpha^2 b_2+\ldots] \tag{11}$$

is introduced where b_1 and b_2 are slowly varying functions of ξ, to account for the period of oscillation which may vary along the length. Substituting from Eqs. (10) and (11), and equating the coefficients of the different powers of α to zero, one obtains

$$\alpha^0: \frac{\partial^4\eta_0}{\partial\xi^4}+B_2\frac{\partial^4\eta_0}{\partial\xi^2\partial\tilde{t}^2}+B_1\frac{\partial^2\eta_0}{\partial\tilde{t}^2}=0 \tag{12a}$$

$$\alpha^1: \frac{\partial^4\eta_1}{\partial\xi^4}+B_2\frac{\partial^4\eta_1}{\partial\xi^2\partial\tilde{t}^2}+B_1\frac{\partial^2\eta_1}{\partial\tilde{t}^2}=-2b_1(\xi)$$

$$\times\left[B_2\frac{\partial^4\eta_0}{\partial\xi^2\partial\tilde{t}^2}+B_1\frac{\partial^2\eta_0}{\partial\tilde{t}^2}\right]+\left(\frac{\partial\eta_0}{\partial\tilde{t}}\right)^2 \tag{12b}$$

$$\alpha^2: \frac{\partial^4\eta_2}{\partial\xi^4}+B_2\frac{\partial^4\eta_2}{\partial\xi^2\partial\tilde{t}^2}+B_1\frac{\partial^2\eta_2}{\partial\tilde{t}^2}=-[2b_2(\xi)$$

$$+b_1^2(\xi)]\left[B_2\frac{\partial^4\eta_0}{\partial\xi^2\partial\tilde{t}^2}+B_1\frac{\partial^2\eta_0}{\partial\tilde{t}^2}\right]-2b_1(\xi)\left[B_2\frac{\partial^4\eta_1}{\partial\xi^2\partial\tilde{t}^2}\right.$$

$$\left.+B_1\frac{\partial^4\eta_1}{\partial\tilde{t}^2}\right]+2b_1(\xi)\left(\frac{\partial\eta_0}{\partial\tilde{t}}\right)^2+2\frac{\partial\eta_0}{\partial\tilde{t}}\frac{\partial\eta_1}{\partial\tilde{t}} \tag{12c}$$

etc.

Equation (12a) is identical in form to Eq. (2) whose solution has been shown to be Eq. (6). However, the rather complex orthogonality condition [Eq. (9)] for the exact mode function does not permit decoupling of the equations later on in the perturbation analysis. Hence, approximate solutions to Eqs. (12) are sought in the form

$$\eta_0=\sum_{r=1}^{\infty}\Phi_r(\xi)f_r(\tilde{t}) \tag{13}$$

where $\Phi_r(\xi)$ are the eigenfunctions of a cantilever beam. Substitution of Eq. (13) into Eqs. (12) gives the solution, up to the first order of approximation, as

$$\eta(\xi,\tilde{t})=\eta_0(\xi,\tilde{t})+\alpha\eta_1(\xi,\tilde{t}) \tag{14}$$

where

$$\eta_0(\xi,t)=\sum_{j=1}^{\infty}A_{0j}\Phi_j(\xi)\cos\bar{\mu}_j^2\tilde{t}$$

and

$$\eta_1(\xi,\tilde{t})=-\frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{\beta_{jkn}\bar{\mu}_j^2\bar{\mu}_k^2A_{0j}A_{0k}\Phi_n(\xi)}{B_1+B_2\sum_{m=1}^{\infty}C_{mn}}\right]$$

$$\times\left\{\frac{[\cos\bar{\mu}_n^2\tilde{t}-\cos(\bar{\mu}_j^2-\bar{\mu}_k^2)\tilde{t}]}{[\bar{\mu}_n^4-(\bar{\mu}_j^2-\bar{\mu}_k^2)^2]}-\frac{[\cos\bar{\mu}_n^2\tilde{t}-\cos(\bar{\mu}_j^2+\bar{\mu}_k^2)\tilde{t}]}{[\bar{\mu}_n^4-(\bar{\mu}_j^2+\bar{\mu}_k^2)^2]}\right\}$$

and

$$A_{0j}=\int_0^1 A_0(\xi)\Phi_j(\xi)\,d\xi$$

$$\bar{\mu}_j^2=\mu_j^2/\left(B_1+B_2\sum_{r=1}^{\infty}C_{rj}\right)^{-1/2}$$

B. Rayleigh-Ritz Method

Consider a uniform cylindrical shell in equilibrium acted upon by static initial stress σ_x^i, σ_θ^i, and $\sigma_{x\theta}^i$. During vibration, the internal stresses in the shell consist of the initial stresses and the additional vibratory stresses σ_x, σ_θ, and $\sigma_{x\theta}$. Assuming there is no interaction between the prestress displacements and the vibratory stresses, the internal strain energy of the shell, taking the prestressed equilibrium state as the reference level, can be written as,[28]

$$U=\tfrac{1}{2}\int_{\text{Vol}}(\sigma_x e_x+\sigma_\theta e_\theta+\sigma_{x\theta}\gamma_{x\theta})\,d(\text{Vol.})$$

$$+\int_{\text{Vol}}(\sigma_x^i e_x+\sigma_\theta^i e_\theta+\sigma_{x\theta}^i\gamma_{x\theta})\,d(\text{Vol})=U_1+U_2 \tag{15}$$

The vibratory stresses σ_x, σ_θ, $\sigma_{x\theta}$ are related to the vibratory strains by Hooke's Law

$$\sigma_x=\frac{E}{1-\nu^2}(e_x+\nu e_\theta) \tag{16a}$$

$$\sigma_\theta=\frac{E}{1-\nu^2}(e_\theta+\nu e_x) \tag{16b}$$

$$\sigma_{x\theta}=\frac{E}{2(1+\nu)}\gamma_{x\theta} \tag{16c}$$

Substituting Eqs. (16), together with the strain-displacement relations of a given shell theory into Eq. (15) and integrating over the thickness, yields the strain energy. Because the initial stresses may be large, it is necessary to use the second-order, nonlinear strain-displacement equations in the U_2 of Eq. (15) while using only the linear relations in U_1. This maintains the proper homogeneity in the orders of magnitude of the terms in the integrands.[28]

U_1 is made up of two parts, one due to stretching (membrane) and the other due to the addition of bending stiffness, i.e.,

$$U_1 = U_{\text{membrane}} + U_{\text{bending}} \tag{17}$$

The membrane component is given by

$$U_{\text{membrane}} = \frac{Eh}{2(1-\nu^2)} \int\int \left[a\left(\frac{\partial x}{\partial \bar{x}}\right)^2 + \frac{1}{a}\left(\frac{\partial y}{\partial \theta} + z\right)^2 + 2\nu \frac{\partial x}{\partial \bar{x}}\left(\frac{\partial y}{\partial \theta} + z\right) + \frac{(1-\nu)}{2a}\left(\frac{\partial x}{\partial \theta} + a\frac{\partial y}{\partial \bar{x}}\right)^2 \right] \mathrm{d}\bar{x}\mathrm{d}\theta \tag{17a}$$

while U_{bending} contains small terms proportional to $(h/a)^3$ that are negligible for modes with small number of circumferential waves. It should be noted that all of the existing shell theories lead to the identical expression for U_{membrane}; the differences occur only in the U_{bending}.

For pressurized tubes, the initial stresses are given by

$$\sigma_x^i = \frac{\sigma_\theta^i}{2} = \frac{pa}{2h} = \frac{N_x}{h}, \qquad \sigma_{x\theta}^i = 0 \tag{18}$$

The strain e of an element at a distance $\bar{z}$ from the middle surface consists of the stretching of the middle surface and that due to rotation of the element. Accordingly,

$$e_x = \epsilon_x + \bar{z}\kappa_x \tag{19a}$$

$$e_\theta = \epsilon_\theta + \bar{z}\kappa_\theta \tag{19b}$$

where ϵ_x, ϵ_θ denote the middle surface strains and κ_x, κ_θ the changes in curvature. Note that ϵ's and κ's are not functions of $\bar{z}$.

The second-order strain-displacement relations according to Washizu's shell theory[31] are

$$\epsilon_x = \frac{\partial x}{\partial \bar{x}} + \frac{1}{2}\left[\left(\frac{\partial x}{\partial \bar{x}}\right)^2 + \left(\frac{\partial y}{\partial \bar{x}}\right)^2 + \left(\frac{\partial z}{\partial \bar{x}}\right)^2\right] \tag{20a}$$

$$\epsilon_\theta = \frac{1}{a}\frac{\partial y}{\partial \theta} + \frac{z}{a} + \frac{1}{2a^2}\left(\frac{\partial z}{\partial \theta}\right)^2 - \frac{y}{a^2}\frac{\partial z}{\partial \theta} \tag{20b}$$

Since the initial stresses are assumed to be due to membrane action, i.e., uniform through the thickness, it is sufficient to

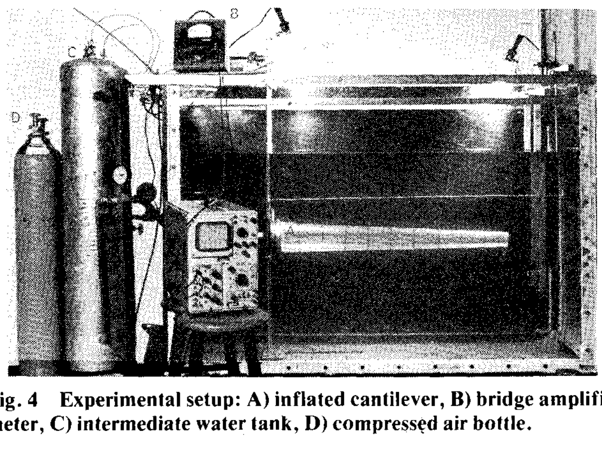

Fig. 4 Experimental setup: A) inflated cantilever, B) bridge amplifier meter, C) intermediate water tank, D) compressed air bottle.

retain only linear terms in the expression relating curvature changes to displacements. There is general agreement among the shell theories for expressions of the middle surface curvatures κ_x and κ_θ, usually taken as

$$\kappa_x = -\frac{\partial^2 a}{\partial \bar{x}^2} \tag{21a}$$

$$\kappa_\theta = \frac{1}{a^2}\frac{\partial y}{\partial \theta} - \frac{1}{a^2}\frac{\partial^2 z}{\partial \theta^2} \tag{21b}$$

Substitution of Eqs. (18-21) into U_2 and integration through the thickness gives

$$\begin{aligned} U_2 = \int\int \Big\{ & N_x\frac{\partial x}{\partial \bar{x}} + \frac{N_x}{2}\left(\frac{\partial x}{\partial \bar{x}}\right)^2 + \frac{N_x}{2}\left(\frac{\partial y}{\partial \bar{x}}\right)^2 + \frac{N_x}{2}\left(\frac{\partial z}{\partial \bar{x}}\right)^2 \\ & - \frac{N_x h}{4}\frac{\partial^2 z}{\partial \bar{x}^2} + \frac{N_\theta}{a}\frac{\partial y}{\partial \theta} + N_\theta\frac{z}{a} + \frac{N_\theta}{2a^2}\left(\frac{\partial x}{\partial \theta}\right)^2 + \frac{N_\theta}{2a^2}\left[\frac{\partial y}{\partial \theta} + z\right]^2 \\ & + \frac{N_\theta}{2a^2}\left[y - \frac{\partial z}{\partial \theta}\right]^2 + \frac{N_\theta h}{4a^2}\frac{\partial y}{\partial \theta} - \frac{N_\theta h}{4a^2}\frac{\partial^2 z}{\partial \theta^2} - \frac{N_\theta h}{4a^2}\frac{\partial y}{\partial \theta} - \frac{N_\theta h}{4a^2}z \\ & - \frac{N_\theta h}{8a^3}\left(\frac{\partial x}{\partial \theta}\right)^2 - \frac{N_\theta h}{8a^3}\left[\frac{\partial y}{\partial \theta} + z\right]^2 - \frac{N_\theta h}{8a^3}\left[y - \frac{\partial z}{\partial \theta}\right]^2 \Big\} a\mathrm{d}\bar{x}\mathrm{d}\theta \end{aligned} \tag{22}$$

The kinetic energy of the inflated shell accounting for the added inertia is

$$\begin{aligned} T = & \frac{1}{2}\rho h \int\int \left[\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2\right] a\mathrm{d}\theta\mathrm{d}\bar{x} \\ & + \frac{1}{2}(1+C_m)\rho_w \int\int\int \left[\frac{\partial y}{\partial t}\sin\theta + \frac{\partial z}{\partial t}\cos\theta\right]^2 r\mathrm{d}\theta\mathrm{d}\bar{x}\mathrm{d}r \\ = & \frac{1}{2}\rho h \int\int \left[\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2 \right. \\ & \left. + \frac{(1+C_m)}{2}\frac{\rho_w}{\rho}\frac{a}{h}\left(\frac{\partial y}{\partial t}\sin\theta + \frac{\partial z}{\partial t}\cos\theta\right)^2\right] a\mathrm{d}\theta\mathrm{d}\bar{x} \end{aligned} \tag{23}$$

The assumed displacements in the beam bending mode are

$$x = [a_1\Phi_r'(\xi) + a_2\Psi_r'(\xi)]\cos\theta\cos\omega t \tag{24a}$$

$$y = [a_3\Phi_r(\xi) + a_4\Psi_r(\xi)]\sin\theta\cos\omega t \tag{24b}$$

$$z = [a_5\Phi_r(\xi) + a_6\Psi_r(\xi)]\cos\theta\cos\omega t \tag{24c}$$

where $\Phi_r(\xi)$ are the characteristic beam functions for cantilevers and

$$\Psi_r(\xi) = \cosh g_r\xi - \cos g_r\xi - \sigma_r(\sinh g_r\xi - \sin g_r\xi)$$

are the characteristic functions for a clamped-pinned beam. The modal forms used in Eqs. (24) are quite realistic, as in practice behavior of an inflated shell suggests boundary conditions between the two sets mentioned in the foregoing.[32]

Substituting the assumed modes [Eq. (24)] into the energy expressions, integrating over the period, and applying the Rayleigh-Ritz procedure, one obtains a sixth-degree frequency equation, which can be rearranged to form an eigenvalue problem of the type

$$[M](a) = \Omega^2[N](a) \tag{25a}$$

where Ω^2 is the dimensionless frequency given by

$$\Omega^2 = \frac{ka^2(1-\nu^2)\omega^2}{E} \tag{25b}$$

The order of the matrices will be $(3n)$ in general, where n is the number of mode shapes in the assumed solution. The system of Eq. (25a) can now be solved by an iteration procedure to obtain the frequencies and mode shapes.

III. Experimental Program

To assess validity of the analysis and to generate relevant design information, an experimental program was undertaken. Model tests were performed in a rectangular water tank (Fig. 4) made of waterproof plywood with front and side plexiglass panels to facilitate observation. A compressed air bottle pressurized an intermediate water tank for inflating a model after the test tank had been filled with water. A pressure gage in the interconnecting piping indicated the inflation pressure.

The test models were made from thin sandwiched films of mylar and polyethylene. Two sheets of commercially available plastic film were pressed together by a heat press. The sandwiched sheet was then wrapped around an appropriate cylindrical blank to form the desired cross section, and the edges sealed with a piece of mylar heat-sealing tape. One end of the tube was closed using a thin plexiglas cap epoxy-glued to the end.

A series of cantilevers of varying radius, length, film thickness, and internal pressure was tested. Natural frequencies were monitored through two waterproof strain gages attached to the top and bottom side of an inflated beam near its root (clamped end). Free vibrations were triggered through initial displacement and release, and the cyclic strain recorded on the oscilloscope indicated the natural frequency. A typical trace on the oscilloscope screen is shown in Fig. 4. In general, the tests for a given setting were repeated at least five times and the average was used. The tests can be repeated with deviations less than 2%.

IV. Results and Discussion

Table 1 compares the experimentally measured frequencies of five inflated cantilevers with various theoretical predictions. Experimental results show a slight increase in frequency with pressure. Although the increase is quite small and almost negligible for all practical purposes, it is significant to recognize that this trend is correctly predicted by the solutions to Flügge's reduced equation [Eq. (1a)]. The difference between the exact frequency [Eq. (8)] and that obtained by mode-approximation is negligible for all cases considered. Both solutions are capable of predicting the natural frequencies with excellent accuracy. The results obtained from the membrane and Herrmann-Armenákas reduced equations are also included in the table for comparison. Both the methods tend to overstress the pressure effects, with the Herrmann-Armenákas theory erroneously predicting a

Table 1 Comparison between analytically and experimentally obtained frequencies, Hz

Tube sizes	Pressure, N/m^2 (psi)	Experimental data	Beam theory	Reduced equations: Membrane	Reduced equations: Herr.-Armen.	Reduced equations: Flügge I[a]	Reduced equations: Flügge II[b]	Rayleigh-Ritz: Washizu 1-term	Rayleigh-Ritz: Washizu 2-term	Rayleigh-Ritz: Membrane 2-term
	0		0.69	0.69	0.69	0.69	0.69	1.05	1.02	1.02
$L = 1.02$ m (40 in.)	2.07×10^4 (3.0)	0.70		1.05	0.20	0.69	0.69	2.40	2.22	1.47
$d = 5.08$ cm (2.0 in.)	3.45×10^4 (5.0)	0.70		1.23	Im.	0.69	0.69	2.97	2.65	1.69
$h = 0.008$ cm (0.003 in.)	4.14×10^4 (6.0)	0.70		1.31	Im.	0.69	0.69	3.22	2.89	1.78
	0		1.04	1.04	1.04	1.05	1.04	1.58	1.54	1.54
$L = 0.91$ m (36 in.)	2.07×10^4 (3.0)	1.06		1.36	0.74	1.05	1.04	2.86	2.71	1.94
$d = 7.62$ cm (3.0 in.)	3.45×10^4 (5.0)	1.07		1.54	0.45	1.05	1.05	3.45	3.19	2.16
$h = 0.008$ cm (0.003 in.)	4.14×10^4 (6.0)	1.08		1.62	0.16	1.06	1.05	3.71	3.39	2.26
	0		1.91	1.91	1.91	1.92	1.91	2.90	2.82	2.82
$L = 0.61$ m (24 in.)	2.07×10^4 (3.0)	2.01		2.32	1.56	1.93	1.91	4.61	4.44	3.34
$d = 5.08$ cm (2.0 in)	3.45×10^4 (5.0)	2.01		2.56	1.28	1.93	1.92	5.45	5.14	3.64
$h = 0.008$ cm (0.003 in.)	4.14×10^4 (6.0)	2.02		2.67	1.12	1.93	1.92	5.82	5.44	3.77
	0		2.34	2.34	2.34	2.38	2.34	3.52	3.41	3.41
$L = 0.61$ m (24 in.)	2.07×10^4 (3.0)	2.38		2.70	2.07	2.39	2.35	4.99	4.82	3.85
$d = 7.62$ cm (3.0 in.)	3.45×10^4 (5.0)	2.39		2.91	1.87	2.39	2.36	5.76	5.48	4.10
$h = 0.008$ cm (0.003 in.)	4.14×10^4 (6.0)	2.39		3.01	1.77	2.39	2.36	6.11	5.76	4.22
	0		3.31	3.31	3.31	3.36	3.31	4.97	4.81	4.81
$L = 0.61$ m (24 in.)	2.07×10^4 (3.0)	3.36		3.57	3.12	3.37	3.32	6.10	5.93	5.13
$d = 7.62$ cm (3.0 in.)	3.45×10^4 (5.0)	3.37		3.73	3.00	3.37	3.32	6.75	6.53	5.33
$h = 0.015$ cm (0.006 in.)	4.14×10^4 (6.0)	3.37		3.81	2.93	3.37	3.32	7.05	6.78	5.43

[a] Exact solution [Eq. (8)]. [b] Mode-approximation solution μ.

decrease in frequency with internal pressures. In a few cases, the frequencies drop to zero and turn imaginary, rendering the validity of the reduced Herrmann-Armenákas equation questionable. The reduced membrane equation predicts a much larger increase in frequency with internal pressure than that observed. On the other hand, the much simpler elementary beam theory, despite its inability to predict the pressure effects on natural frequencies, yields results of reasonable accuracy.

Also shown in Table 1 are the theoretical predictions based on the Rayleigh-Ritz method for the Washizu and the membrane shell theories. For the Washizu theory, results for one- and two-term approximations are shown for comparison. Their agreement with experimental results are poor, and the large difference between the one- and two-term approximations indicates a slow convergence of the natural frequencies. The fact that the convergence of results can be very slow has been observed by other investigators such as Sewall and Naumann,[22] and Resnick and Dugundji.[23] Sewall and Naumann compared analytical frequencies with experimental results for clamped-free shells without prestress. They used seven terms in the assumed mode shapes to obtain convergence of the Ritz procedure. Resnick and Dugundji, using an energy approach, found that theory and experiment agreed only for modes with more than five circumferential waves. Thus, a large number of terms will be required to converge to the right value, but the amount of algebra involved is great since the order of the governing matrices increases rapidly with the number of terms used in the approximation.

V. Concluding Remarks

The significant conclusions based on the free vibration analysis can be summarized as follows:

1) For the cylindrical inflated cantilevers vibrating in the beam-bending mode, the governing three-dimensional shell equations do not permit simple solutions. Although an exact procedure is available, it has been sparingly applied because of the great amount of work required. Numerical and approximate techniques have mostly been used in the shell vibration studies to date. For the present study, it is found that Flügge's shell equation in reduced form is capable of predicting the vibrational behavior of uniform cylindrical beams subjected to internal pressure. Accurate predictions are possible even with the approximate solution of the equation discussed here. However, the reduction technique should be applied with care, since various shell theories give results which may be significantly different. The reduced equations for the membrane and Hermann-Armenákas theory fail to give reasonable results. It should also be noted that for certain shell theories, the equations are nonsymmetric (e.g., the Timoshenko-Voss equations used by Fung et al.[33]). In these cases, the reduction process will not give meaningful results.

2) The elementary beam theory gives predictions of reasonable accuracy although the theory does not incorporate internal pressure effects. Fortunately, the effect of pressure, at least in the range investigated here ($<4.14\times10^4$ N/m^2 or 6.0 psi) appears to be insignificant. Even with the internal pressure of 4.14×10^4 N/m^2 or 6.0 psi ($P=0.02$), the increase in frequency would amount to less than 2%.

3) The Rayleigh-Ritz method does not give accurate results in the present investigation. The convergence of the results is slow and the relatively great amount of work required to achieve acceptable accuracy cannot be justified.

4) The hydrodynamic drag damping causes only an amplitude decay and does not affect the resonant frequencies of the cantilevers up to the first-order approximation.

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